Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

stract, and can, therefore, never define the whole truth. For the whole truth of things is always individual, and is never expressible in terms merely of abstraction, nor in terms of merely logical implication. On the other hand, as soon as you consider any individual fact, as, for instance, the fact that this man has this form of intuition, you consider what, if true, is no topic of mathematical science. I conclude, then, that Kant's theory of the basis of mathematics has been in one respect wholly abandoned, and properly so, by the modern logic of mathematics. In another respect, precisely in so far as Kant declared that constructive synthesis and observation of its ideal results are both necessary for mathematics, Kant was unquestionably right. And as nobody before him had so clearly seen this fact, and as the progress of mathematical logic since his time has been so profoundly influenced by his criticisms, we owe to him an enormous advance in our reflective insight in this field.

JOSIAH ROYCE.

HARVARD UNIVERSITY.

SOME OUTSTANDING PROBLEMS FOR PHILOSOPHY

FOR a long space of time the domains of philosophy and mathematics were regarded less as intersecting spheres of interest than as adjacent fields separated by a kind of 'dead line' that no worker in either might venture to cross without risking the loss both of his identity and of the respect of his fellows. I once heard a distinguished mathematician say that the study of mathematics acts on the metaphysical instinct like sulphur on the itch. Undoubtedly that savant had a philosophy, but, like the Irishman's snake, he was unconscious of it, and he was the less tolerant on that account. On the other hand, the mathematician has not infrequently been compelled to forgive such disrespect as that of Sir Wm. Hamilton's on ground analogous to the good old Catholic principle of invincible ignorance. Happily, the tokens more and more abound that the uncanny day of such misunderstandings is rapidly passing away. It may return again, but not for some generations. A new era has begun that shall be distinguished by intellectual sympathy and cooperation, by increasing wholesomeness of scholarship. The indicia have reference to old records. They indicate the reestablishment of broken traditions of an older time when philosopher and mathematician were often united in a single personality. Such men as C. S. Peirce and Pearson and Mach and Couturat and Poincaré and Georg Cantor, exemplify clearly enough that the larger incarnations are not impossible even in these crowded days of narrow specialization.

Nowhere else is the community of interest so vast and conspicuous as in the theory of assemblages. By the latter I mean the doctrine variously entitled *Mengenlehre* and *Mannigfaltigkeitslehre* by the Germans, *théorie des ensembles* by the French, and often referred to in English as the theory of manifolds or aggregates, or by other analogous designations. Many of its ideas date at least as ancient as historical thought, and have figured in important ways in logic, in philosophy and in mathematics steadily from the earliest times. On the other hand, many of its chief concepts, its characteristic and ruling notions, and their organization into a distinct and self-supporting body of coherent doctrine, may be said to constitute the latest great mathematico-philosophical creation. If mathematics is the most fundamental of the sciences, assemblage theory is destined to be regarded as the most fundamental branch of mathematics. Herein lies its immeasurable import for philosophy. Viewed in retrospect, it appears as an inevitable product of the modern critical spirit. Already it is seen to underlie and interpenetrate both geometry and analysis. Its connection with modern logic is most intimate, often approximating identity with the latter. And philosophy in some wide-awake quarters has already recognized in the doctrine of manifolds her own most promising and inviting field.

Now in this young, vigorous and rapidly unfolding doctrine, there have been for some time certain recognized outstanding questions that ought to challenge mathematician and philosopher alike, in equal measure certainly, if not in manner the same. These problems are of two classes: those that have been very recently solved, and those which still await solution. I purpose to state some of the problems here in the hope that some rising philosopher may feel their challenge and, in case of the latter class, direct a yet unconquered genius to their solution.

A question of the former class is: *Can every assemblage be well-ordered?* It has just received an affirmative answer by E. Zermelo.[1] The fundamental character of the problem, the simplicity of the means employed in its solution and especially the bearings of the solution on kindred dependent questions, justify a brief report upon the matter in this place.

What, then, is an assemblage? Any collection of objects or things (elements) of whatsoever kind or kinds is an assemblage. An assemblage, to be mathematically available, to be available, that is, for the purposes of rigorous thought, must be *defined;* and it is said to be defined when and only when we know enough about it to know that an arbitrarily given object either is or is not an element of it. When is a defined assemblage said to be well-ordered (*wohlgeordnet, bien defini*)? The answer is, when and only when its elements have been so disposed in fact or in thought that one may affirm of it the following three propositions: (1) of any two of its elements a and b, one, as a, *precedes, i. e.*, is of *lower rank*, and the other, as b, *comes after, i. e.*, is of *higher rank;* (2) of any triplet of its elements a, b, c, if a is of lower rank than b, and b is lower than c, then a is lower than c; (3) the assemblage has an element of *lowest* rank, a *first* element, and the same is true of *every part* of the assemblage; that is, of every assemblage whose elements are elements of the given assemblage.

---

[1] *Mathematische Annalen*, Band 59, Heft 4, November, 1904.

The simplest possible example of a well-ordered assemblage is that of the integers in their natural order $1, 2, 3, \cdots, \nu, \nu+1, \cdots$, it being agreed that any *element* (figure) ν shall be of *lower* rank than an element $\nu+\sigma$ if the *number* ν is *less* than the *number* $\nu+\sigma$. It is easily seen that the assemblage in question satisfies the three necessary and sufficient conditions. The same is true of the assemblage of all *rational* numbers greater than *zero* and less than 1, if it be agreed that of every two such numbers $\frac{a}{b}$ and $\frac{c}{d}$, that one shall have lower rank which has the smaller number for sum of its terms, and that, if $a+b=c+d$ in case of two numbers $\frac{a}{b}$ and $\frac{c}{d}$, then the smaller of these shall have the lower rank. Thus arranged, the assemblage stands: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{1}{5}, \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \cdots$. But if the same elements be arranged in their so-called natural order, that is, on a line of unit-length from zero to 1, and if it be agreed that of two numbers that one shall be of lower rank whose distance from the origin (zero) is the less, then, though the assemblage is indeed *ordered,* it is *not well-ordered,* for in its present arrangement it fails to satisfy condition (3); it has, for example, no *first* element, no element of lowest rank, and the like is true of countless numbers of its parts. It is plain, then, that not every assemblage, not even every ordered assemblage, is well-ordered. What Zermelo has established is that every assemblage *can be* well-ordered.

To such as know that it is impossible to *denumerate* the points of a continuum, that is, to set up a one-to-one correspondence between, say, the points of a unit-line and the integers, Zermelo's result is apt to be surprising, and, because of its far-reaching implications, it is really astonishing. Of such implications, two of the most important and striking are these: *first,* every conceivable assemblage can be so arranged that each element (unless it be the last, an insignificant special case) shall be followed by a *next;* and *second,* that every assemblage can be so arranged that *every* sequence of its elements a, b, c, $\cdots$ in which a has higher rank than b, b higher rank than c, etc., must have an *end,* a last element. Any one who will seriously attempt to conceive the points of the room in which he is sitting so arranged that the assemblage shall have the two properties just stated, can not fail to have a deepening sense of the marvelous character, the awful comprehensiveness, of Zermelo's proposition.

And what, pray, is the stuff of which the demonstration is woven? What are the auxiliary concepts employed? Few indeed and simple, though tenuous and slippery and lithe. First is the notion of the assemblage of *all* the *parts* of a given assemblage E. Another idea is that of associating or pairing with each of those parts some definite element of the part, to be then known as the 'distinguished' (*ausgezeichnete*) element of its associate part. A third concept is that of the *off-cut* of a well-ordered assemblage, the term denoting the part (of a well-ordered assemblage) whose elements, all of them, are of lower rank than a given element of the assemblage. Obviously to each element of the assemblage there corresponds a definite off-cut. A very important rôle is played by the notion of *γ-assemblage:* an assemblage A is a γ-assemblage A provided it is well-ordered and, in addition, possesses the property that, if e is an arbitrary element of A and if O is the off-cut corresponding to e, then e is always the 'distinguished' element of $E - O$, where the latter symbol denotes all the elements of E except those of O. Such are the *Sommerfäden* out of which the author has contrived the beautiful figure of his demonstration.

I have already pointed out two of its important bearings. Another of its consequences is that it immediately removes from the field of doubt or controversy another question of the utmost fundamental significance. I refer to the question concerning the *comparability* of every pair of assemblages. The meaning of the question can be made sufficiently clear by a word of explanation. Two assemblages, A and B, are said to be *equivalent* when and only when it is possible to set up between the elements of A and those of B a one-to-one correspondence. The simplest familiar example of such equivalence is that where A denotes the assemblage of integers and B that of their doubles. Such examples abound in the worlds of space and number. On the other hand, A is said to be *less* than B and B greater than A, when A and B satisfy the two conditions: (1) A has no (proper)[2] part equivalent to B; (2) B has a (proper) part equivalent to A. Now either of the relations, equivalence and less (greater), excludes the other. That admits of rigorous demonstration. But herewith is not resolved the question whether one or other of the relations subsists between every two assemblages. If A and B are equivalent, then neither is less than the other; if one of them is less than the other, the two are not equivalent; but herein is plainly no ground for inference that they must be *either* equivalent *or* one less than the other. Possibly in some cases neither relation holds, *i. e.,* A and B may not be comparable. In such an eventuality, we should have to recognize two classes of assemblages: the comparable and the non-comparable. Behold the matter from another point of view (due to Borel). If A and B be two assemblages, then the following four cases may arise: (1) A has a part equivalent to B but B none equivalent to A; (2) B has a part equivalent to A but A none equivalent to B; (3) A has a part equivalent to B and B has a part equivalent to A; (4) A has no part equivalent to B and B none equivalent to A. The symmetry here is noteworthy. In passing, I wish to signalize the importance of the principle of symmetry in fundamental questions of logic, and to suggest the topic, the logical significance of esthetic principles, or its equivalent, as possibly worthy of an aspirant to the doctorate. To resume, it is noteworthy that the four cases are not merely hypothetical, they *exist.* Cases (1) and (2) are, of course, essentially the same, and are exemplified by taking for A or B a point continuum and for B or A the totality of integers. In such case one is less than the other. To exemplify case (3) it suffices to let A stand for the assemblage of real (rational and irrational) numbers and to let B denote the assemblage of the irrational numbers. It has been proved that if A and B fall under case (3), they are equivalent. Case (4) alone remains. That it is not *empty* is shown at once by letting A denote the assemblage a_1, a_2 and B the assemblage b_1, b_2. In this *particular* example, A and B are indeed equivalent, but does the notion of equivalence attach to *every* pair A and B of (4), a case shown to exist? Georg Cantor, easily the primate of all contributors to *Mengenlehre,* believed the answer should be affirmative. But he had not been able to give a scientific, *i. e.,* a *transferable,*[3] demonstration, and men of the type of Borel held the matter in doubt. Was Cantor's conviction *rooted* in esthetic soil? Was it due to a fine commanding sense of what esthetically ought to be? At all events his conviction was just, if not logically justified, for it is an obvious corollary to Zermelo's proposition that every two assemblages are comparable and that, if they belong to case (4), they are equivalent.

---

[2] A proper part of A is a part composed of some but not all of the elements of A.

---

[3] Concepts and proofs are essentially *social* affairs. They must be intelligible to at least two minds, or, what is tantamount, to one person at least twice. Such need not be true of emotion.

So much for 'problems' that may be regarded as very recently settled. I turn now to the second kind. Of these the most conspicuous is that of determining whether there is thinkable an assemblage having a power intermediate to that of the denumberable assemblage and that of the continuum. These terms demand explanation. If the assemblages A and B are *equivalent,* they are said to have the *same power* (*Mächtigkeit*). An assemblage having the same power as the assemblage $a_1, a_2, a_3, \cdots, a_\nu, \cdots$, without end, is said to be denumerable, or to have the power of the denumerable assemblage. To avoid confusion, it is well to note that *power* has not been defined; it is *sameness of power* that has been defined. An assemblage, such as that of the irrational numbers, that has the same power as that of the real numbers, or the points of a line-segment, is said to have the *power of the continuum.* Now it is well known and readily admits of proof that, if A has the power of the denumerable assemblage and B the power of the continuum, A and B have *not* the same power. In fact, as above noted, A is, in the case supposed, less than B. Is there an assemblage I intermediate to such an A and B? That means, is there an I greater than A and less than B? All efforts to prove or disprove that such is the case have been thus far baffled. The proof offered by the late Paul Tannery has been found defective. The problem meets one on the very threshold of assemblage theory and *apparently* demands acumen and ingenuity rather than a large fund of mathematical knowledge.

It may be useful to view the question from another point of view. As is well known, Cantor has defined 'classes' (I), (II), $\cdots$ of higher and higher power, where by *higher* is here meant that (I) regarded as an assemblage is less than (II) similarly regarded, and so on. This achievement of his, which is one of the boldest and most brilliant in the annals of mathematic genius, was done as follows. He made use of three *principles of number generation.* These, stated in full generality, are: (1) the addition of unity, 1, to a number already formed; (2) positing, in case of an endless sequence of numbers (integers) having no greatest, a new number that shall be the first integer after the sequence and immediately greater than every number of the sequence; (3) subjecting all numbers generable by (1) and (2) to the condition that all those preceding any specified one of them shall have the same power as a class already defined.

Starting with 1, class (I), $1, 2, \cdots, \nu, \cdots$, is defined by (1). By (2), ω is put to be first after (I) and greater than every number of (I). Combination of (1) and (2) yields $\omega + 1$, $\omega + 2$, $\cdots$, 2ω,

$2\omega + 2, \cdots, \omega^2, \omega^2 + 1, \cdots, n\omega^2, \cdots, n\omega^\omega, \cdots$. Class (II) is struck out by application of (3) to the foregoing sequence of transfinites. Cantor proves that (I) and (II) are not of the same power and that there is no assemblage between (I) and (II) in the sense of intermediate above explained. Hence our problem is to show that (II) has the power of the continuum or that it has not. It is interesting to note that with exquisite impropriety Cantor has denominated his three principles *Logical Moments.* Moments they undoubtedly are; but logical, they are just as undoubtedly not. Neither are they illogical. They are super-logical, being precisely the critical points in his process of transfinite creation where logic pauses for a *new start,* the adoption of a *principium,* a choice, never compelled, but only solicited and always declinable. The possibility, the spiritual vision, of a generalization, logic may and often does present, but the deed of generalization logic is impotent to do or force.

I shall close by stating another outstanding question that is immediately suggested by the procedure just described, though the same question is encountered on many another path. It is: Is there in the *constitution* of the human mind any limitation to the definition of higher and higher powers of assemblages? Is, in other words, an assemblage of all assemblages and elements logically available? Is the notion meaningless? Is it free from interior contradiction? Would such an assemblage have to be an element of (in) itself? Whatever the answers be, can we, do we, ever or always, dispense with such a 'notion'? Can the word *all* be safely used *conjunctively*—in the sense of none excluded?

CASSIUS J. KEYSER.

COLUMBIA UNIVERSITY.

REVIEWS AND ABSTRACTS OF LITERATURE

Elements of Metaphysics. A. E. TAYLOR. London, Methuen & Co.; New York, The Macmillan Co. 1903. Pp. xvi + 419.

This book is the most comprehensive and systematic treatise on metaphysics that we have yet had from a living writer in English. It discusses with definiteness and clearness of style and in well-ordered arrangement the chief problems of ultimate philosophy. Professor Taylor is a disciple, though not a slavish follower, of Mr. Bradley, and he endeavors in a constructive and original fashion to unite the general speculative position of Mr. Bradley with the interpretations of actual experience which have found fullest expression and elucidation in the writings of Professors Ward and Royce. And whatever one's opinion as to the final coherence of the various strains in Professor Taylor's doctrine one must